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On the absolute instability of semi-implicit schemes for hydrostatic models

Andrei Bourchtein*, Ludmila Bourchtein

Institute of Physics and Mathematics, Pelotas State University, Rua Anchieta 4715 bloco K, ap.304, Pelotas 96015-420, Brazil

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Abstract

The dependence of the linear stability of two-time-level finite-difference semi-implicit schemes on the choice of reference temperature profile is studied. Particular vertical profiles of the temperature are considered to derive analytical conditions of stability. Analysis is made for general form of different model parameters such as the number of vertical levels and their distribution, the time step size, and the values of the viscosity coefficients. The derived conditions of stability are more restrictive than those for three-time-level schemes, but obtained necessary and sufficient condition for constant vertical lapse rates of the temperature has the form frequently applied to three-time-level schemes: the basic temperature profile should be warmer than the actual one. Performed numerical experiments show that the last restriction is neither necessary nor sufficient condition of stability for general temperature profiles.

MSC: 65M12; 76M20

Keywords: Atmospheric hydrostatic model; Semi-implicit scheme; Linear stability

1. Introduction

The atmospheric models are based on the Navier–Stokes equations or some simplified versions of these equations. The most useful simplification is the hydrostatic model, which allows to filter insignificant acoustic waves. Even so, the hydrostatic equations support the relatively slow synoptic processes carrying the main part of available energy (advection and Rossby waves) as well the fast gravity waves with secondary contribution to atmospheric dynamics. The stiffness and nonlinearity of the mathematical models of atmospheric dynamics have direct effect on the choice of the numerical methods used for computation of approximated solutions to these models. On the one hand, explicit schemes are rarely employed because of excessive restriction on time step size reflecting presence of the fast gravity waves. On the other hand, the fully implicit schemes are not used due to complexity of nonlinear systems arising at each time step. Therefore, the most popular approach to construction of numerical schemes is semi-implicit time approximation with explicit treatment of main nonlinear terms and implicit discretization of secondary linear terms responsible for gravity waves.

* Corresponding author. Tel./fax: +55 53 32757343 .
E-mail address: burstein@terra.com.br (A. Bourchtein).

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It allows the use of greater time steps and reducing the implicit part of the scheme to solution of linear systems. Besides, only the linear terms with constant coefficients should be approximated implicitly in order to ensure an efficient solution of the equations for implicit part. The semi-Lagrangian semi-implicit approach was shown to be the most efficient method as applied to hydrostatic atmospheric models because it allows to circumvent the Courant stability condition with respect to both gravity and advective waves, and ensures rather extended time steps, about 1 h, which are physically justified [10,19].

There are different physically reasonable ways to choose the terms to be treated implicitly. However, the constructed scheme may be absolutely unstable, that is, unstable for any time step. Since early applications of semi-implicit schemes in the multi-level atmospheric models the phenomenon of absolute instability discovered by Burridge [6] attracted attention of the researches [2,3,7,16–18]. The essence of the problem consists of appearance of instability in the part of equations responsible for fast gravity waves, which are approximated implicitly. It was shown that this behavior is caused by explicit treatment of the deviations from the reference vertical temperature profile. It is important to keep the explicit approximation of these deviations in order to maintain relatively simple structure of the implicit equations at each time step. Therefore, various numerical experiments and theoretical analyses were made to clarify how to avoid this instability. It was discovered numerically in [6] that instability does not appear if one chooses the temperature of the reference profile warmer than the actual one. Simmons et al. [16] presented first analytical results using continuous in vertical formulation of the hydrostatic primitive equations. For isothermal reference and actual temperature profiles they showed that "stability occurs provided only that the actual temperature at any height does not exceed the reference temperature at that height by more than a factor of 2" [16]. They complemented the analytical study by numerical evaluations of stability in the context of a finite-difference scheme and revealed that in some cases instability occurs even for small positive differences between actual and reference profiles. Therefore, the proposed remedy, widely adopted by numerical modelers, was to use rather warm isothermal reference temperature in semi-implicit schemes. Côté et al. [7] extended analytical study to the case of non-isothermal profiles in the context of a finite-element vertical discretization and found that stability of semi-implicit scheme can be ensured if the actual static stability is less than double the reference static stability. All these analyses were concerned with three-time-level models because they were more popular in atmospheric modeling until the 1990s.

In the late 1980s it was shown that two-time-level schemes can support rather simple design of the three-time-level models and still ensure more accurate solutions as applied to shallow water equations [13,20]. Therefore, since the 1990s different atmospheric centers start to adopt two-time-level baroclinic schemes and some tests were employed to reveal that a similar condition of a warmer reference temperature guarantees absolute stability of the gravity waves [8,12,14,21]. Also more specific studies were performed following the analytical-numerical framework defined in [16]. In particular, Simmons and Temperton [18] analyzed classical two-time-level semi-implicit differencing with finite-difference approximation in vertical. They found that in order to obtain a stable scheme the reference temperature should be warmer than the actual one, that is, two times more rigid constraint as compared with three-time level schemes. Cullen [9] examined predictor–corrector semi-implicit scheme and showed that additional implicit iteration can increase the range of stability for two-time-level scheme up to that of three-time-level scheme. Bénard [2] proposed a general analytical framework for study of the absolute instability suitable for both the hydrostatic primitive equations and the fully compressible Euler equations. His approach is based on space-continuous formulation of the governing equations for unbounded atmosphere and some additional assumptions related to possibility of transformation of the bounded atmosphere systems in unbounded ones, stability of the stationary states and separability of equations for individual normal modes. In particular, this approach was applied to non-extrapolating semi-implicit schemes for the Euler equations with isothermal atmosphere and it was shown that non-extrapolating two-time-level semi-implicit differencing is unstable, but additional implicit iterations can recover the scheme stability [2]. In order to enhance stability of semi-implicit schemes, Bénard proposed in [3] to remove the vertical thermal advective term from the terms approximated implicitly and showed that this modified version of semi-implicit time differencing is more stable than classical one when applied to three-time-level schemes in the context of the hydrostatic primitive equations.

It is worth to note that due to complexity of the characteristic equations for amplification factors, complete analytical solution of the stability problem has not been reached neither for three- nor for two-time-level schemes. Therefore, all studies of the absolute instability, including this research, follow the general framework defined in [16]: combination of exact analysis for some specific cases with numerical evaluations for more general situations. The aim of our study is to clarify some stability properties of the classical two-time-level semi-implicit schemes in the context of the hydrostatic primitive equations, which remain to be the most used differential system in the numerical weather prediction and atmospheric modeling. We apply von Neumann analysis of computational stability for the case of horizontally uniform reference temperature, and for vertically discretized equations.

Let us make some remarks on comparison between vertically discrete and continuous approaches. Despite of some attractive points of the general approach by Bénard, we think that analysis of the vertically discretized equations in bounded atmosphere is more suitable for our purposes. First, we formulate the hydrostatic equations in the pressure-based coordinates, whose different versions are mostly used in numerical atmospheric models. The treatment of the vertical boundary conditions in the pressure-based coordinates is not straightforward in the Bénard approach, leading to necessity of application of additional linear operators to the governing hydrostatic equations [2]. Second, continuous approach is not suitable for treating the multilayered systems (e.g., when the reference and actual temperature profiles have distinct lapse rates in different layers of the atmosphere) [3]. Since our primarily goal is to advance with analytical treatment of the stability problem as far as possible (appealing to numerical evaluations only in the most complicated cases), we try to keep the form of the governing equations as simple as possible in order to obtain more treatable characteristic equations for time-dependent amplitudes of unknown functions. Besides, all numerical models are designed for the bounded atmosphere and discrete vertical representation can be made as accurate as we need simply increasing the number of levels and managing its distribution. These are the principal arguments that stimulate the use of vertically discrete framework.

Application of the spectral analysis of computational stability to the vertically discretized hydrostatic equations allows us to derive a more treatable (although still rather complex) form of the polynomial characteristic equation for the amplification factors. It makes possible analytical solution for such special cases as the case of large time steps and proportionality between actual and reference vertical structure matrices. In particular, we have obtained the stability criterion in the terms of the vertical lapse rates of the reference and actual profiles, which is more rigid than a similar constraint derived in [4,7] for three-time-level schemes. In the case of specific vertical grids, we have shown that the obtained criterion is necessary and sufficient for two-time-level schemes, while it is only sufficient (but not necessary) condition for three-time-level differencing. The last relation between stability of two- and three-time-level schemes is consistent with the results presented in [18]. However, in the case of arbitrary distribution of the vertical levels (essentially, in the case of the increased resolution in the upper atmosphere) the stability conditions for two- and three-time-level schemes are virtually coincide.

The paper is structured as follows. In Section 2, we present analysis of the simplified model equations, which helps to reveal some characteristics of the studied instability. In Section 3, a semi-implicit scheme for the linearized hydrostatic equations is presented and von Neumann analysis of the linear stability is employed to derive the characteristic equation for amplification factors. Some special cases of the characteristic equation are considered and solved analytically in Section 4. In particular, a necessary and sufficient condition of stability for temperature profiles with constant lapse rates is derived. In Section 5, we study the effect of modification to semi-implicit discretization proposed by Bénard on stability of two-time-level schemes. The results of numerical experiments presented in Section 6 reveal the stability properties of two-time-level schemes for more general situations. In particular, it is shown that instability can be observed when basic temperature is warmer than the actual one and that the scheme can be stable for small time steps when conditions of stability for large time steps are violated. The final remarks are presented in the last section.

2. Semi-implicit discretization and stability of the model equations

To clarify the mechanism of the absolute instability, let us start with a simple system of two-dimensional gravity waves subject to viscous forces in shallow atmosphere:

$$D_t = -\nabla^2 \Phi + \alpha \nabla^2 D, \tag{2.1}$$

$$\Phi_t = -cD + \beta \nabla^2 \Phi. \tag{2.2}$$

Here t is the time, x and y are the Cartesian coordinates, $D = u_x + v_y$ is the horizontal divergence, u, v are the components of the wind velocity, $\Phi = gz$ is the geopotential, g is the gravitational acceleration, z is the height of the pressure surface, $c \approx 300\,\mathrm{m/s}$ is the propagation velocity of the gravity waves, α and β are the viscosity coefficients (simulating turbulence effects or numerical dissipation), and $\nabla^2 \equiv \partial_{xx} + \partial_{yy}$ is the horizontal Laplace operator. This system can be obtained by linearizing the barotropic (or shallow water) equations about a state of rest and neglecting the rotating effect.

To simulate the semi-implicit discretization in real three-dimensional models we consider the following second-order time discretization of (2.1)–(2.2):

$$\frac{D^{n+1}-D^n}{\tau}-\alpha\nabla^2\frac{D^{n+1}+D^n}{2}=-\nabla^2\frac{\Phi^{n+1}+\Phi^n}{2}, \tag{2.3}$$

$$\frac{\Phi^{n+1}-\Phi^n}{\tau}-\beta\nabla^2\frac{\Phi^{n+1}+\Phi^n}{2}=-c\frac{D^{n+1}+D^n}{2}-(c-\bar{c})\frac{3D^n-D^{n-1}}{2}, \tag{2.4}$$

where τ is the time step and superscripts $n-1$, n and $n+1$ denote the values at the "old" $(n-1)\tau$, "current" $n\tau$ and "new" $(n+1)\tau$ time levels, respectively. The introduced parameter $\bar{c}$ is a specific constant generally different from c and its physical meaning will be clear later in exposition of hydrostatic model. Meanwhile one can consider that c is a spatial function weakly varying about its mean value $\bar{c}$. In this case, the used separation of terms in the second equation allows a simplification of the implicit part without loss of accuracy. The scheme (2.3)–(2.4) represents the Crank–Nicholson approximation for all terms, except for the last term in the second equation, which is approximated in the Adams–Bashforth mode. Since the last term is an additional one, in atmospheric modeling the scheme (2.3)–(2.4) is usually referred to as two-time-level scheme, whose design is quite different from three-time-level time approximation applied to all terms of the differential model [13,14,20,21]. In this text we will follow this terminology. The scheme (2.3)–(2.4) is a simple prototype of a popular pattern of discretization of gravity waves in the atmospheric models [8,12,21].

To apply von Neumann stability analysis, we consider an individual wave

$$\begin{pmatrix} D \\ \Phi \end{pmatrix}^n (x,y)=\begin{pmatrix} E \\ H \end{pmatrix}\mu^n\cdot\exp(\mathrm{i}m_x x+\mathrm{i}m_y y) \tag{2.5}$$

in expansion of unknown functions in Fourier series with respect to variables x and y. Here, E, H, μ are the amplitudes and amplification factor of the Fourier harmonic with wave numbers (m_x, m_y). Substituting (2.5) in (2.3) and (2.4), we obtain linear algebraic homogeneous system for the amplitudes, which allows non-trivial solution if, and only if, its determinant is zero. This gives rise to characteristic equation, which defines the amplification factor as a function of other scheme parameters:

$$P(\mu)=\det\begin{pmatrix} \mu-1+\frac{\tau}{2}\alpha m^2(\mu+1) & -\frac{\tau}{2}m^2(\mu+1) \\ \frac{\tau}{2}\bar{c}(\mu^2+\mu)+\frac{\tau}{2}(c-\bar{c})(3\mu-1) & \mu^2-\mu+\frac{\tau}{2}\beta m^2(\mu^2+\mu) \end{pmatrix}=0,$$

where $m^2=m_x^2+m_y^2$. Denoting $\theta=m^2\tau/2$ and calculating determinant we obtain the following third-order polynomial equation

$$P(\mu)=\mu[\mu-1+\theta\alpha(\mu+1)][\mu-1+\theta\beta(\mu+1)]+\theta\frac{\tau}{2}(\mu+1)[\bar{c}\mu(\mu+1)+(c-\bar{c})(3\mu-1)]=0. \tag{2.6}$$

According to von Neumann criterion, the scheme (2.3), (2.4) is stable if all roots of the polynomial $P(\mu)$ lie on the unit circle. To evaluate when this condition holds, it is suitable to change the variable μ by formula

$$z=\frac{\mu+1}{\mu-1},$$

which represents the one-to-one mapping between the unit circle and the left half-plane with the boundary of the unit circle mapped onto the imaginary axis. After some algebra Eq. (2.6) can be rewritten in the form

$$Q(z)\equiv Az^3+Bz^2+Cz+D=(z+1)(1+\theta\alpha z)(1+\theta\beta z)+\theta\frac{\tau}{2}z[\bar{c}z(z+1)+(c-\bar{c})(z+2)(z-1)]=0.$$

According to the Lienard–Shippard criterion [11], all roots of this polynomial equation have negative real part if, and only if, all the coefficients A, B, C, D and determinant $\Delta = BC - AD$ are positive:

$$A = \theta^2\alpha\beta + \theta\frac{\tau}{2}c > 0,$$

$$B = \theta\alpha + \theta\beta + \theta^2\alpha\beta + \theta\frac{\tau}{2}c > 0,$$

$$C = 1 + \theta\alpha + \theta\beta - \theta\tau(c - \bar{c}) > 0,$$

$$D = 1 > 0,$$

$$BC - AD = \theta\alpha + \theta\beta + \left(\theta\alpha + \theta\beta + \theta^2\alpha\beta + \theta\frac{\tau}{2}c\right)(\theta\alpha + \theta\beta - \theta\tau(c - \bar{c})) > 0.$$

Therefore, one can conclude that $Q(z)$ is the Schur polynomial (i.e., all its roots are within the unit circle) for arbitrary $\alpha, \beta \geqslant 0$, if and only if, $0 < c < \bar{c}$.

Obviously, the first inequality holds for barotropic or shallow water equations and means that the primitive system (2.1), (2.2) is of the hyperbolic type. Otherwise, the primitive equations do not admit correct initial value problem and, consequently, its numerical solution through (2.3), (2.4) has no meaning. Therefore, positiveness of the parameter c is the consequence of the physical collocation of the differential problem and it does not add additional requirement for numerical scheme. Another situation is observed for second inequality: this is the actual requirement resulting from used numerical approximations (2.3)–(2.4). To complete this analysis, we observe that including the unit modulus roots is equivalent to considering the equality $c = \bar{c}$. Thus, the essential stability condition of the scheme (2.3), (2.4) has the form $c \leqslant \bar{c}$. If this condition does not hold, then the presented semi-implicit approximation will be unstable for any time step.

3. Semi-implicit two-time-level hydrostatic scheme and characteristic equation

The system for three-dimensional gravity waves can be obtained by linearization of the hydrostatic equations of non-rotating atmosphere about a state of rest

$$\hat{u} \equiv \hat{v} \equiv \hat{\omega} \equiv 0, \quad \hat{T} = \hat{T}(p), \quad \hat{\Phi} = \hat{\Phi}(p) : \hat{\Phi}_{\ln p} = -R\hat{T}.$$

Using pressure vertical coordinate one can write the linearized hydrostatic equations as follows [10]:

$$D_t = -\nabla^2\Phi + \alpha\nabla^2 D, \tag{3.1}$$

$$\Phi_{\ln p} = -RT, \tag{3.2}$$

$$D + \omega_p = 0, \tag{3.3}$$

$$T_t = \frac{R\hat{T}(\Gamma_d - \hat{\Gamma})}{gp}\omega + \beta\nabla^2 T. \tag{3.4}$$

Besides the parameters introduced in Section 2, the following notations are used: p is the pressure, $\omega \equiv \mathrm{d}p/\mathrm{d}t$ is the vertical component of velocity, T is the temperature, R is the gas constant, c_p is the specific heat at constant pressure, $\Gamma_d \equiv g/c_\mathrm{p}$ is the adiabatic lapse rate and $\hat{\Gamma} \equiv (gp/R\hat{T})\hat{T}_\mathrm{p} = -\hat{T}_z$ is the vertical lapse rate of the temperature profile $\hat{T}$. Essential vertical boundary conditions have the following linearized form

$$\omega(p_{\mathrm{up}}) = 0, \quad (\Phi_t - R\hat{T}\omega - \beta\nabla^2\Phi)(p_{\mathrm{lw}}) = 0, \tag{3.5}$$

where p_{up} and p_{lw} are upper and lower pressure levels, respectively.

Using the hydrostatic equation (3.2) to substitute Φ for T in (3.4), integrating the obtained equation with respect to p from p to p_{lw} and integrating (3.3) with respect to p from p_{up} to p, we can rewrite the system (3.1)–(3.4) in the form

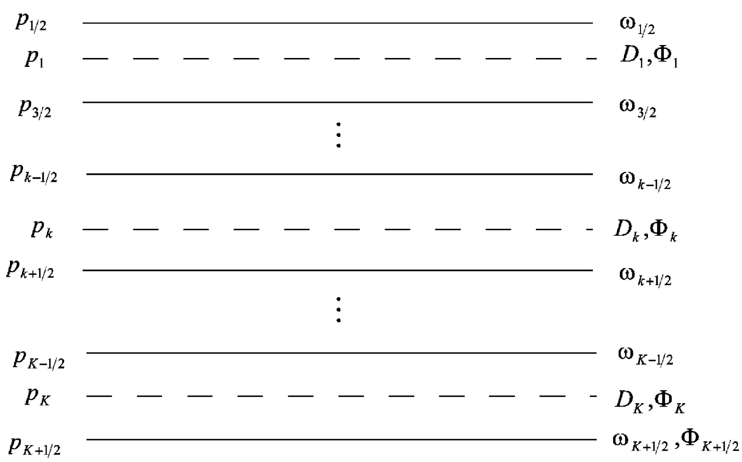

Fig. 1. The Lorenz staggered vertical grid.

suitable for subsequent vertical discretization used in numerical schemes

$$(\partial_t - \alpha\nabla^2)D = -\nabla^2\Phi, \tag{3.6}$$

$$\omega = -\int_{p_{\mathrm{up}}}^{p} D\,\mathrm{d}p, \tag{3.7}$$

$$(\partial_t - \beta\nabla^2)\Phi = \frac{R^2}{g}\int_p^{p_{\mathrm{lw}}} \frac{\hat{T}(\Gamma_d - \hat{\Gamma})}{p}\omega\,\mathrm{d}(\ln p) + R\hat{T}(p_{\mathrm{lw}})\omega(p_{\mathrm{lw}}). \tag{3.8}$$

Introducing the most popular Lorenz staggered vertical grid [1] (see Fig. 1), which divides the considered atmosphere in K vertical layers with boundaries $p_{k+1/2}, k = 0, \ldots, K$

$$p_{\mathrm{up}} = p_{1/2} < p_{3/2} < \cdots < p_{k-1/2} < p_{k+1/2} < \cdots < p_{K-1/2} < p_{K+1/2} = p_{\mathrm{lw}}, \tag{3.9}$$

and with inner levels $p_k, k = 1, \ldots, K$ satisfying the natural inequalities

$$p_{k-1/2} < p_k < p_{k+1/2}, \quad k = 1, \ldots, K, \tag{3.10}$$

we can discretize Eqs. (3.6)–(3.8) as follows:

$$(\partial_t - \alpha\nabla^2)D_k = -\nabla^2\Phi_k, \quad k = 1, \ldots, K; \tag{3.11}$$

$$\omega_{k+1/2} = -\sum_{i=1}^{k} D_i(p_{i+1/2} - p_{i-1/2}), \quad k = 1, \ldots, K; \quad D_i = (u_x + v_y)_i; \tag{3.12}$$

$$\begin{aligned}(\partial_t - \beta\nabla^2)\Phi_k = R\hat{T}_{K+1/2}\omega_{K+1/2} + \frac{R^2}{g}\Bigg[\sum_{i=k+1}^{K} \frac{\hat{T}_{i-1/2}(\Gamma_d - \hat{\Gamma}_{i-1/2})}{p_{i-1/2}}\omega_{i-1/2} \\ \times \ln\frac{p_i}{p_{i-1}} + \frac{\hat{T}_{K+1/2}(\Gamma_d - \hat{\Gamma}_{K+1/2})}{p_{K+1/2}}\omega_{K+1/2}\ln\frac{p_{K+1/2}}{p_K}\Bigg], \quad k = 1, \ldots, K.\end{aligned} \tag{3.13}$$

In the last formula, a summation is defined to be zero if the lower limit of the summation index exceeds the upper limit.

Using the K order vectors

$$\mathbf{D} = (D_1, \ldots, D_K)^{\mathrm{T}}, \quad \mathbf{\Phi} = (\Phi_1, \ldots, \Phi_K)^{\mathrm{T}}, \quad \omega = (\omega_{3/2}, \ldots, \omega_{K+1/2})^{\mathrm{T}},$$

we can rewrite (3.11)–(3.13) in the form

$$(\partial_t - \alpha\nabla^2)\mathbf{D} = -\nabla^2\mathbf{\Phi}, \tag{3.14}$$
$$\boldsymbol{\omega} = -\mathbf{B}\mathbf{D}, \tag{3.15}$$
$$(\partial_t - \beta\nabla^2)\mathbf{\Phi} = \hat{\mathbf{A}}\boldsymbol{\omega}. \tag{3.16}$$

Here, $\hat{\mathbf{A}}$ and $\mathbf{B}$ are the $K \times K$ upper and lower triangular matrices with elements

$$a_{j,k} = \hat{a}_k = \frac{R^2}{g}\frac{\hat{T}_{k+1/2}(\Gamma_d - \hat{\Gamma}_{k+1/2})}{p_{k+1/2}} \ln\frac{p_{k+1}}{p_k}, \quad k = 1, \ldots, K-1, \quad j \leqslant k; \quad a_{j,k} = 0, \quad j > k; \tag{3.17}$$
$$a_{K,K} = \hat{a}_K = \frac{R^2}{g}\frac{\hat{T}_{K+1/2}(\Gamma_d - \hat{\Gamma}_{K+1/2})}{p_{K+1/2}} \ln\frac{p_{K+1/2}}{p_K} + R\hat{T}_{K+1/2};$$
$$b_{j,k} = b_k = p_{k+1/2} - p_{k-1/2}, \quad k = 1, \ldots, K, \quad j \geqslant k; \quad b_{j,k} = 0, \quad j < k. \tag{3.18}$$

Note that the elements a_k depend on the temperature profile and therefore matrix $\hat{\mathbf{A}}$ can be considered as a value of matrix function of temperature $\mathbf{A}(T)$ at $T = \hat{T}$ in such a way that $\hat{\mathbf{A}} = \mathbf{A}(\hat{T})$.

Substituting $\mathbf{D}$ for $\boldsymbol{\omega}$ in (3.16) we reduce the system (3.14)–(3.16) to a simpler form

$$(\partial_t - \alpha\nabla^2)\mathbf{D} = -\nabla^2\mathbf{\Phi}, \tag{3.19}$$
$$(\partial_t - \beta\nabla^2)\mathbf{\Phi} = -\hat{\mathbf{C}}\mathbf{D}, \tag{3.20}$$

where $\hat{\mathbf{C}} = \hat{\mathbf{A}}\mathbf{B}$ is the vertical structure matrix, which depends on the chosen temperature profile and vertical discretization. For different vertical discretizations it can be shown that $\hat{\mathbf{C}}$ is oscillatory matrix and, therefore, all its eigenvalues are real and positive [5]. This is essential property of the matrix $\hat{\mathbf{C}}$ used in the following analysis of linear stability.

To keep the essence of semi-implicit time differencing in atmospheric models we represent the actual temperature profile $\hat{T}(p)$ in the form $\hat{T} = \bar{T} + \tilde{T}$, where $\bar{T}$ is basic (or reference) profile and $\tilde{T}$ is its deviation. The only term of the system (3.19), (3.20) that depends on $\hat{T}$ is the matrix $\hat{\mathbf{C}}$ on the right-hand side of equation (3.20). Therefore, we represent $\hat{\mathbf{C}} = \bar{\mathbf{C}} + \tilde{\mathbf{C}}$, where $\bar{\mathbf{C}} \equiv \mathbf{C}(\bar{T}) = \mathbf{A}(\bar{T})\mathbf{B} \equiv \bar{\mathbf{A}}\mathbf{B}$ is the basic matrix and $\tilde{\mathbf{C}}$ is the deviation matrix. Matrix $\hat{\mathbf{C}}$ is called the full or actual matrix. In this paper we will consider only the cases when the actual and basic temperature profiles are statically stable, that is, $\hat{\Gamma} < \Gamma_d$ and $\bar{\Gamma} < \Gamma_d$. Otherwise, the primitive differential problem is not well posed. These conditions imply non-negativity of the matrices $\hat{\mathbf{A}}$ and $\bar{\mathbf{A}}$ and, consequently, $\hat{\mathbf{C}}$ and $\bar{\mathbf{C}}$.

According to semi-implicit time discretization the basic matrix term is approximated implicitly and the deviation is extrapolated explicitly:

$$\frac{\mathbf{D}^{n+1} - \mathbf{D}^n}{\tau} - \alpha\nabla^2\frac{\mathbf{D}^{n+1} + \mathbf{D}^n}{2} = -\nabla^2\frac{\mathbf{\Phi}^{n+1} + \mathbf{\Phi}^n}{2}, \tag{3.21}$$
$$\frac{\mathbf{\Phi}^{n+1} - \mathbf{\Phi}^n}{\tau} - \beta\nabla^2\frac{\mathbf{\Phi}^{n+1} + \mathbf{\Phi}^n}{2} = -\bar{\mathbf{C}}\frac{\mathbf{D}^{n+1} + \mathbf{D}^n}{2} - (\hat{\mathbf{C}} - \bar{\mathbf{C}})\frac{3\mathbf{D}^n - \mathbf{D}^{n-1}}{2}. \tag{3.22}$$

We will call this type of time differencing the classical semi-implicit scheme. Here, as before, τ is the time step and superscripts $n-1$, n and $n+1$ denote the values at the "old" $(n-1)\tau$, "current" $n\tau$ and "new" $(n+1)\tau$ time levels, respectively. In this way, we arrive to the system similar to previously considered discretization (2.3)–(2.4) for the linearized shallow water equations. Of course, the three-dimensional scheme is more general, and the shallow water discretization can be obtained from this scheme if we set the number of the vertical levels K to be equal to 1. Therefore, it is not surprising that for some choices of the actual and basic matrices ($\hat{\mathbf{C}}$ and $\bar{\mathbf{C}}$) the time discretized schemes (3.21)–(3.22) could be unstable. In what follows, we will describe some situations when it happens and what could be done to prevent these situations.

According to von Neumann method of stability analysis we consider particular solution in the wave form

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{\Phi}\end{pmatrix}^n(x,y) = \begin{pmatrix}\mathbf{W}\\ \mathbf{H}\end{pmatrix}\cdot\mu^n\cdot\exp(\mathrm{i}m_x x + \mathrm{i}m_y y),$$

where K-order vectors $\mathbf{W}$, $\mathbf{H}$ describe the vertical structure of the amplitudes of the individual wave with the wave numbers (m_x, m_y) and μ is the amplification factor describing behavior of the amplitudes with respect to time. For stability of the numerical scheme the amplification factors should lie on the unit circle for any pair of the wave numbers. Substituting this wave solution in (3.21)–(3.22), we obtain the linear algebraic system for the vectors $\mathbf{W}$, $\mathbf{H}$:

$$\frac{\mu-1}{\tau}\mathbf{W}+\alpha m^2\frac{\mu+1}{2}\mathbf{W}=m^2\frac{\mu+1}{2}\mathbf{H}, \tag{3.23}$$

$$\frac{\mu^2-\mu}{\tau}\mathbf{H}+\beta m^2\frac{\mu^2+\mu}{2}\mathbf{H}=-\bar{\mathbf{C}}\frac{\mu^2+\mu}{2}\mathbf{W}-(\hat{\mathbf{C}}-\bar{\mathbf{C}})\frac{3\mu-1}{2}\mathbf{W}, \tag{3.24}$$

where $m^2=m_x^2+m_y^2\geqslant 0$. The system (3.23)–(3.24) has non-trivial solution if, and only if, its determinant is equal to zero. Simplifying expression for the determinant we obtain the following characteristic equation:

$$\begin{aligned}\det\ Q &\equiv \det\begin{pmatrix}[2(\mu-1)+\tau\alpha m^2(\mu+1)]\mathbf{I} & -\tau m^2(\mu+1)\mathbf{I}\\ \tau(\mu^2+\mu)\bar{\mathbf{C}}+\tau(3\mu-1)(\hat{\mathbf{C}}-\bar{\mathbf{C}}) & [2(\mu^2-\mu)+\tau\beta m^2(\mu^2+\mu)]\mathbf{I}\end{pmatrix}\\ &=\det\{\lfloor(\mu^2+\mu)\bar{\mathbf{C}}+(3\mu-1)(\hat{\mathbf{C}}-\bar{\mathbf{C}})\rfloor\tau^2m^2(\mu+1)+[2(\mu-1)\\ &\quad+\tau\alpha m^2(\mu+1)][2(\mu-1)+\tau\beta m^2(\mu+1)]\mu\mathbf{I}\}=0.\end{aligned} \tag{3.25}$$

This is polynomial equation of order $3K$, where K is the number of the vertical levels.

Note that the same kind of equation will be obtained if any commonly used space discretization is applied to (3.21)–(3.22). For example, applying the central difference approximations on space grid A with mesh size h [15] and searching for solution in the form of discrete wave with discrete wave numbers m_x, m_y, we obtain Eqs. (3.23)–(3.24) with m substituted by

$$m_A^2=\frac{4}{h^2}(\sin^2 m_xh+\sin^2 m_yh).$$

If one uses staggered grid C [15], then the only modification is again expression for m:

$$m_C^2=\frac{1}{h^2}\left(\sin^2\frac{m_xh}{2}+\sin^2\frac{m_yh}{2}\right).$$

Similar results can be obtained for any type of the regular space grid with central difference approximation of the derivatives. Therefore, semi-discrete system (3.21)–(3.22) keeps all properties of the completely discrete approximations.

4. Analytical solutions to classical scheme

4.1. No deviations

If there are no deviations from the basic temperature, that is, $\hat{\mathbf{C}}=\bar{\mathbf{C}}$, then Eq. (3.25) has K-fold zero root. Since that root satisfies stability condition, we can suppose that $\mu\neq 0$ to rewrite (3.25) in the form

$$\det\{\bar{\mathbf{C}}-\lambda\mathbf{I}\}=0,$$

where

$$\lambda=-\frac{[2(\mu-1)+\tau\alpha m^2(\mu+1)][2(\mu-1)+\tau\beta m^2(\mu+1)]}{\tau^2m^2(\mu+1)^2} \tag{4.1}$$

are the positive eigenvalues of the matrix $\bar{\mathbf{C}}$ ranging from 0 to 10^5 under usual temperature conditions. Introducing a new unknown x by formula

$$x=\frac{\mu-1}{\mu+1},$$

whose inverse is

$$\mu = \frac{1+x}{1-x},$$

we rewrite (4.1) in the form

$$4x^2 + 2x\tau m^2(\alpha+\beta) + \tau^2 m^2(\lambda + \alpha\beta m^2) = 0.$$

Solving the last equation for x we obtain

$$x_\pm = \frac{\tau m}{4}\left[-m(\alpha+\beta) \pm \sqrt{m^2(\alpha-\beta)^2 - 4\lambda}\right].$$

If $m^2(\alpha-\beta)^2 - 4\lambda < 0$, then the roots are complex conjugate $x_\pm = a \pm \mathrm{i}b$ with $a \leqslant 0$ and evaluation of $|\mu|$ gives

$$|\mu|^2 = \left|\frac{1+(a\pm \mathrm{i}b)}{1-(a\pm \mathrm{i}b)}\right|^2 = \frac{(1+a)^2+b^2}{(1-a)^2+b^2} \leqslant 1.$$

If $m^2(\alpha-\beta)^2 - 4\lambda \geqslant 0$, then both roots are non-positive, and consequently,

$$|\mu| = \left|\frac{1+x_\pm}{1-x_\pm}\right| \leqslant 1.$$

Thus, all roots μ lie on the unit circle because all eigenvalues of the matrix $\bar{\mathbf{C}}$ are positive. Therefore, we obtain well-known result of the Crank–Nicholson discretization: if all terms in (3.19)–(3.20) are treated implicitly, then the scheme (3.21)–(3.22) is absolutely stable.

4.2. Large time steps

Considering the limiting form of (3.25) as τ approaches infinity we obtain

$$\det\{\lfloor(\mu^2+\mu)\bar{\mathbf{C}} + (3\mu-1)(\hat{\mathbf{C}} - \bar{\mathbf{C}})\rfloor m^2(\mu+1) + \alpha\beta m^4(\mu+1)(\mu^2+\mu)\mathbf{I}\} = 0. \tag{4.2}$$

First, we note that $\mu = -1$ is K-fold root, which does not violate the stability of the scheme. Therefore (4.2) can be simplified to the form

$$\det\{\lfloor(\mu^2+\mu)\bar{\mathbf{C}} + (3\mu-1)(\hat{\mathbf{C}} - \bar{\mathbf{C}})\rfloor + \alpha\beta m^2(\mu^2+\mu)\mathbf{I}\} = 0. \tag{4.3}$$

Since (4.3) is still too hard for exact analysis, we consider the case $\alpha\beta = 0$. Then (4.3) assumes the form

$$\det\lfloor(\mu^2+\mu)\bar{\mathbf{C}} + (3\mu-1)(\hat{\mathbf{C}} - \bar{\mathbf{C}})\rfloor = 0.$$

Due to definition of the matrices $\bar{\mathbf{C}}$ and $\hat{\mathbf{C}}$ we have

$$\det\lfloor(\mu^2+\mu)\bar{\mathbf{A}} + (3\mu-1)(\hat{\mathbf{A}} - \bar{\mathbf{A}})\rfloor \cdot \det \mathbf{B} = 0.$$

Since $\det \mathbf{B} \neq 0$ and $\bar{\mathbf{A}}$ and $\hat{\mathbf{A}}$ are upper triangular matrices, the last equation transforms to

$$\prod_{k=1}^{K}[(\mu^2+\mu)\bar{a}_k + (3\mu-1)(\hat{a}_k - \bar{a}_k)] = 0. \tag{4.4}$$

The solutions of (4.4) are

$$\mu_{k\pm} = \frac{1}{2}\left[2 - 3d_k \pm \sqrt{9d_k^2 - 8d_k}\right], \quad d_k = \frac{\hat{a}_k}{\bar{a}_k} > 0, \quad k = 1, \ldots, K.$$

If $9d_k^2 - 8d_k \leqslant 0$, that is, $d_k \leqslant \frac{8}{9}$, then

$$|\mu_{k\pm}|^2 = 1 - d_k < 1.$$

If $d_k > \frac{8}{9}$, then $|\mu_{k-}| > |\mu_{k+}|$ and solution of inequality $|\mu_{k-}| \leqslant 1$ shows that the last is true if, and only if, $d_k \leqslant 1$. Joining two considered evaluations, we obtain that the scheme is stable in the limiting case of large time steps if, and only if,

$$0 < d_k \leqslant 1, \quad k = 1, \dots, K,$$

that is,

$$0 < \hat{a}_k \leqslant \bar{a}_k, \quad k = 1, \dots, K. \tag{4.5}$$

According to (3.17) the last inequality can be rewritten as follows:

$$\hat{T}_{k+1/2}(\Gamma_d - \hat{\Gamma}_{k+1/2}) \leqslant \bar{T}_{k+1/2}(\Gamma_d - \bar{\Gamma}_{k+1/2}), \quad k = 1, \dots, K-1; \tag{4.6}$$

$$\hat{T}_{K+1/2}(\Gamma_d - \hat{\Gamma}_{K+1/2} + \xi_{K+1/2}) \leqslant \bar{T}_{K+1/2}(\Gamma_d - \bar{\Gamma}_{K+1/2} + \xi_{K+1/2}),$$

$$\xi_{K+1/2} = \left(\frac{1}{p_{K+1/2}} \ln \frac{p_{K+1/2}}{p_K} \right)^{-1} \cdot \frac{g}{R}. \tag{4.7}$$

If the actual and basic temperature profiles have constant lapse rates $\hat{\Gamma}$ and $\bar{\Gamma}$, then

$$\hat{T}_{k+1/2} = \hat{T}_{K+1/2} \left(\frac{p_{k+1/2}}{p_{K+1/2}} \right)^{R\hat{\Gamma}/g}, \quad \bar{T}_{k+1/2} = \bar{T}_{K+1/2} \left(\frac{p_{k+1/2}}{p_{K+1/2}} \right)^{R\bar{\Gamma}/g}, \quad k = 1, \dots, K-1 \tag{4.8}$$

and conditions (4.6) take the form

$$\left(\frac{p_{k+1/2}}{p_{K+1/2}} \right)^{R(\hat{\Gamma}-\bar{\Gamma})/g} \leqslant \frac{\bar{T}_{K+1/2}(\Gamma_d - \bar{\Gamma})}{\hat{T}_{K+1/2}(\Gamma_d - \hat{\Gamma})}, \quad k = 1, \dots, K-1. \tag{4.9}$$

If $\bar{T}_{K+1/2} = \hat{T}_{K+1/2}$, then (4.9) is equivalent to condition

$$\bar{\Gamma} \leqslant \hat{\Gamma}. \tag{4.10}$$

Let us note that the obtained stability conditions (4.6)–(4.7) and their simplified form (4.9) are two times more restrictive with respect to the choice of the basic temperature profile than those for three-time-level schemes [4,7]. In particular, in the case of the specific vertical grid the condition (4.10) for constant lapse rates is necessary and sufficient for two-time-level models, while it is sufficient but not necessary one for three-time-level models. It agrees with the conclusions of Simmons and Temperton [18], which found the stability condition for two-time-level schemes in the form $\hat{T} \leqslant \bar{T}$ instead of $\hat{T} \leqslant 2\bar{T}$ for three-time-level schemes.

Let us also note that condition (4.10) is necessary and sufficient not only when $\bar{T}_{K+1/2} = \hat{T}_{K+1/2}$, but also when $\bar{T}_{K+1/2} > \hat{T}_{K+1/2}$ if one considers arbitrary vertical discretization. In fact, if (4.10) does not hold, then the function in the left-hand side of (4.9) has a negative exponent and, consequently, there exists a level with a small pressure $p_{k+1/2}$ such that inequality (4.9) is violated, which implies instability. On the other hand, if one deals with specific vertical grid, then the smallest pressure value $p_{1/2}$ is fixed and one can find the lapse rates $\hat{\Gamma}$ and $\bar{\Gamma}$, which do not satisfy (4.10), but condition (4.9) holds. Due to the same reasons, (4.10) becomes necessary and sufficient condition for both two- and three-time-level models in the case of arbitrary distribution of the vertical levels, while for specific vertical grid (4.10) is not necessary condition for three-time-level schemes.

4.3. Proportional matrices

If $\hat{\mathbf{C}} = v\bar{\mathbf{C}}$ then the characteristic equation (3.25) is reduced to

$$\det\{[\mu^2 - 1 + \alpha m^2 \tau(\mu^2 + 1)][\mu^2 - 1 + \beta m^2 \tau(\mu^2 + 1)]\mathbf{I} + m^2 \tau^2 (\mu + 1) \lfloor (\mu - 1)^2 + v(3\mu - 1) \rfloor \bar{\mathbf{C}}\} = 0. \tag{4.11}$$

Since we consider only the statically stable actual and basic temperature profiles, the coefficient v must be positive. The case $v = 1$ was solved above, so we assume that $v \neq 1$.

Using parameter λ defined by relation

$$\lambda\tau^2m^2(\mu+1)[\mu^2+\mu+(v-1)(3\mu-1)]=-[2(\mu-1)+\tau\alpha m^2(\mu+1)][2(\mu-1)+\tau\beta m^2(\mu+1)]\mu, \quad (4.12)$$

Eq. (4.10) can be rewritten as follows:

$$\det\{\bar{\mathbf{C}}-\lambda\mathbf{I}\}=0,$$

that is, λ are positive eigenvalues of the matrix $\bar{\mathbf{C}}$ [5].

Let us consider Eq. (4.12) for unknown μ. It is suitable to introduce new unknown z by formula

$$\mu=\frac{z+1}{z-1},$$

which is one-to-one transformation between the unit circle and the left half-plane of the complex plane. In this way we transform Eq. (4.12) to the form

$$P(z)\equiv Az^3+Bz^2+Cz+D=(2+\tau\alpha m^2z)(2+\tau\beta m^2z)(z+1)+\mu\tau^2m^2z(vz^2+vz-2v+2)=0. \quad (4.13)$$

According to the Lienard–Shippard criterion [11], the following inequalities should be satisfied:

$$A=\mu v\tau^2m^2+\tau^2m^4\alpha\beta>0,\quad B=2\tau m^2(\alpha+\beta)+\tau^2m^4\alpha\beta+\mu v\tau^2m^2>0,\quad D=4>0, \quad (4.14)$$
$$C=4+2\tau m^2(\alpha+\beta)-2\mu(v-1)\tau^2m^2>0, \quad (4.15)$$

$$\begin{aligned}\Delta&=2\tau m^2(\alpha+\beta)(4+2\tau m^2(\alpha+\beta)-2\mu\tau^2m^2(v-1))\\&\quad+(\tau^2m^4\alpha\beta+\mu v\tau^2m^2)(2\tau m^2(\alpha+\beta)-2\mu\tau^2m^2(v-1))>0.\end{aligned} \quad (4.16)$$

Evidently, inequalities (4.14) hold. Inequality (4.15) results in

$$v<v_0=1+\frac{2+\tau m^2(\alpha+\beta)}{\mu\tau^2m^2},$$

and (4.16) gives

$$v<v_1 \quad (4.17)$$

with

$$v_1=\frac{1}{2\mu\tau^2m}\left[\tau m(\mu\tau-(\alpha+\beta)-\tau m^2\alpha\beta)+\sqrt{\tau^2m^2(\mu\tau+3(\alpha+\beta)+\tau m^2\alpha\beta)^2+16\tau(\alpha+\beta)}\right].$$

It can be shown that $v_1<v_0$ and for great values of μ or small values of $(\alpha+\beta)$ inequality (4.17) reduces to $v\leqslant 1$.

Since $\bar{\mathbf{C}}=\bar{\mathbf{A}}\mathbf{B}$ and $\hat{\mathbf{C}}=\hat{\mathbf{A}}\mathbf{B}$, assumption $\hat{\mathbf{C}}=v\bar{\mathbf{C}}$ is equivalent to $\hat{\mathbf{A}}=v\bar{\mathbf{A}}$, that is, $\hat{a}_k=v\bar{a}_k$, $k=1,\dots,K$. By applying (3.17) these relations can be expressed as follows:

$$\hat{T}_{k+1/2}(\Gamma_d-\hat{\Gamma}_{k+1/2})=v\bar{T}_{k+1/2}(\Gamma_d-\bar{\Gamma}_{k+1/2}),\quad k=1,\dots,K-1;$$
$$\hat{T}_{K+1/2}(\Gamma_d-\hat{\Gamma}_{K+1/2}+\xi_{K+1/2})=v\bar{T}_{K+1/2}(\Gamma_d-\bar{\Gamma}_{K+1/2}+\xi_{K+1/2}),$$

where $\xi_{K+1/2}$ is specified in (4.7). Since stability condition is $v\leqslant 1$, one can observe that the last two relations coincide with inequalities (4.6) and (4.7) and, consequently, they have a similar interpretation.

5. Modified scheme

Following the modification proposed by Bénard [3], let us consider time differencing slightly different from classical schemes (3.21)–(3.22). The principal idea is to exclude the thermal advective term, which contribute to instability of the scheme, from the basic terms, that is, from the terms treated implicitly. To this end, the matrix $\hat{\mathbf{A}}$ defined in (3.17) should be split in two parts $\hat{\mathbf{A}} = \hat{\mathbf{A}}_E + \hat{\mathbf{A}}_F$ with non-zero entries

$$\begin{aligned} e_{j,k} &= \frac{R^2}{g}\frac{\hat{T}_{k+1/2}\Gamma_d}{p_{k+1/2}}\ln\frac{p_{k+1}}{p_k}, \quad k=1,\ldots,K-1, \quad j \leqslant k; \\ e_{K,K} &= \frac{R^2}{g}\frac{\hat{T}_{K+1/2}\Gamma_d}{p_{K+1/2}}\ln\frac{p_{K+1/2}}{p_K} + R\hat{T}_{K+1/2}; \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} f_{j,k} &= -\frac{R^2}{g}\frac{\hat{T}_{k+1/2}\hat{\Gamma}_{k+1/2}}{p_{k+1/2}}\ln\frac{p_{k+1}}{p_k}, \quad k=1,\ldots,K-1, \quad j \leqslant k; \\ f_{K,K} &= -\frac{R^2}{g}\frac{\hat{T}_{K+1/2}\hat{\Gamma}_{K+1/2}}{p_{K+1/2}}\ln\frac{p_{K+1/2}}{p_K}, \end{aligned} \tag{5.2}$$

respectively. Then the matrix $\hat{\mathbf{C}}$ introduced in system (3.19)–(3.20) can be partitioned in $\hat{\mathbf{C}} = \hat{\mathbf{C}}_E + \hat{\mathbf{C}}_F$, $\hat{\mathbf{C}}_E = \hat{\mathbf{A}}_E\mathbf{B}$, and $\hat{\mathbf{C}}_F = \hat{\mathbf{A}}_F\mathbf{B}$. Proceeding in a similar manner as in deriving system (3.21)–(3.22), we represent $\hat{\mathbf{C}}_E = \bar{\mathbf{C}}_E + \tilde{\mathbf{C}}_E$ with $\bar{\mathbf{C}}_E = \bar{\mathbf{A}}_E\mathbf{B}$ and apply implicit time differencing only to the terms with $\bar{\mathbf{C}}_E$:

$$\frac{\mathbf{D}^{n+1}-\mathbf{D}^n}{\tau} - \alpha\nabla^2\frac{\mathbf{D}^{n+1}+\mathbf{D}^n}{2} = -\nabla^2\frac{\mathbf{\Phi}^{n+1}+\mathbf{\Phi}^n}{2}, \tag{5.3}$$

$$\frac{\mathbf{\Phi}^{n+1}-\mathbf{\Phi}^n}{\tau} - \beta\nabla^2\frac{\mathbf{\Phi}^{n+1}+\mathbf{\Phi}^n}{2} = -\bar{\mathbf{C}}_E\frac{\mathbf{D}^{n+1}+\mathbf{D}^n}{2} - (\hat{\mathbf{C}}-\bar{\mathbf{C}}_E)\frac{3\mathbf{D}^n-\mathbf{D}^{n-1}}{2}. \tag{5.4}$$

We will refer to the last system as the modified scheme. Evidently, the characteristic equation (3.25) will be substituted by a similar one with matrix $\bar{\mathbf{C}}_E$ in place of $\bar{\mathbf{C}}$. Applying the large time step approximation, one can readily obtain the stability criterion

$$0 < \hat{a}_k \leqslant \bar{e}_k, \quad k=1,\ldots,K \tag{5.5}$$

instead of (4.5). Respectively, conditions (4.6)–(4.7) are changed to

$$\hat{T}_{k+1/2}(\Gamma_d - \hat{\Gamma}_{k+1/2}) \leqslant \bar{T}_{k+1/2}\Gamma_d, \quad k=1,\ldots,K-1; \tag{5.6}$$

$$\hat{T}_{K+1/2}(\Gamma_d - \hat{\Gamma}_{K+1/2} + \xi_{K+1/2}) \leqslant \bar{T}_{K+1/2}(\Gamma_d + \xi_{K+1/2}), \tag{5.7}$$

and in the case of the constant lapse rates one obtains

$$\left(\frac{p_{k+1/2}}{p_{K+1/2}}\right)^{R(\hat{\Gamma}-\bar{\Gamma})/g} \leqslant \frac{\Gamma_d}{\Gamma_d - \hat{\Gamma}}, \quad k=1,\ldots,K-1 \tag{5.8}$$

under assumption $\bar{T}_{K+1/2} = \hat{T}_{K+1/2}$.

Evidently, the right-hand side in (5.8) is greater than that in (4.9), but it is not great enough to obtain a restriction softer than (4.10) for the classical scheme in the cases when pressure levels in the upper atmosphere are involved.

Let us advance a little further with comparison between the stability of the classical and the modified schemes. As we can see the criterion (5.6)–(5.7) is a softer constraint on the choice of the basic temperature than (4.6)–(4.7). In fact, for the basic and actual profiles with constant lapse rates, $\bar{\Gamma} > \hat{\Gamma}$ results in unstable classical scheme (3.21)–(3.22), while the modified scheme (5.3)–(5.4) allows alleviation of this restriction. For example, assuming $\bar{T}_{K+1/2} = \hat{T}_{K+1/2} = 285\,\mathrm{K}$, we can show that, according to criterion (5.8) for large time steps and numerical evaluations (not presented here), the stability of the modified scheme is preserved for all time steps when $\bar{\Gamma} = 0.0065\,\mathrm{K/m} > \hat{\Gamma} = 0.0011\,\mathrm{K/m}$ in the atmosphere layer between 500 hPa and 1000 hPa, and when $\bar{\Gamma} = 0.0065\,\mathrm{K/m} > \hat{\Gamma} = 0.0020\,\mathrm{K/m}$ in the atmosphere layer between 200 hPa and 1000 hPa. However, if we consider the atmosphere above these layers, the overall stability can be easily broken, for example, extending two profiles with the same basic and actual lapse rates or using isothermal actual profile above the indicated layers. In the last case, unconditional instability is inevitable whatever the basic profile is. For large time steps it can be seen from conditions (5.6), which assume the form $\hat{T}_{k+1/2} \leqslant \bar{T}_{k+1/2}$ equivalent to that of the classical scheme, and for different time steps it follows from performed numerical experiments (not shown here). Unfortunately, this situation is quite natural for the real temperature profiles when considered layer of the atmosphere includes tropopause (with nearly isothermal profile) and stratosphere (with negative lapse rates), which is the usual situation in the current numerical models. For example, characteristic analytical profile with constant lapse rate in troposphere and isothermal stratosphere considered in [3,16] provides the stable modified scheme only if the level of the actual tropopause is above the level of the reference tropopause, or if the model atmosphere does not include the stratosphere layer. Otherwise, the modified scheme is absolutely unstable just like the classical one.

Besides a weak stability in the case of nearly isothermal actual profiles, the modified scheme contains at least two more potential sources of concern, both related to the scheme accuracy. First, approximation of one of the leading terms in the primitive system is supposed to be done with the highest accuracy chosen for the main terms. However, in the modified scheme (5.3)–(5.4) the central time averaging is used for all leading terms, except for the extrapolated thermal convection term. Formally, both approximations are of the second order of accuracy, but truncation error of the latter is six times greater than of the former. Second, it is known from practice of numerical modeling, that better accuracy of the scheme is frequently achieved when compensating terms in the equations are approximated in a similar manner (e.g., [14]), which is not a case of the modified scheme. So it may happen that an increase in stability for the modified scheme is achieved by means of deterioration of accuracy. Of course, all these points should be carefully verified in real numerical models in order to weight all the “pros and cons” of this approach.

6. Numerical evaluations

In this section we present the results of computation of amplification factor for the cases, which are hard to treat analytically. In all the experiments described below we used two-level vertical grid with $p_{5/2}=1000\,\text{hPa}$, $p_2=750\,\text{hPa}$, $p_{3/2}=500\,\text{hPa}$, and $p_1=250\,\text{hPa}$.

First, let us consider non-constant lapse rates. In this case, condition (4.10) can be violated in some vertical layers still giving the stable numerical scheme. For example, choosing the actual and basic temperature profiles as follows:

$$\bar{T}_{5/2}=273\,\text{K},\quad \bar{T}_{3/2}=241.69\,\text{K},\quad \bar{\Gamma}_{5/2}=\bar{\Gamma}_{3/2}=0.006\,\text{K/m}.$$
$$\hat{T}_{5/2}=273\,\text{K},\quad \hat{T}_{3/2}=234.95\,\text{K},\quad \hat{\Gamma}_{5/2}=0.0095\,\text{K/m},\quad \hat{\Gamma}_{3/2}=0.0059\,\text{K/m},$$

one obtains the absolutely stable scheme. The results of the respective computations are presented in Fig. 2, where the modulus of the amplification factor is drawn as a function of time step. Four different curves correspond to different values of the viscosity coefficients: $\alpha=\beta=0$, solid curve; $\alpha=\beta=1$, dashed curve; $\alpha=\beta=10$, dot-dashed curve; and $\alpha=\beta=100$, dotted curve. Hereinafter, the values of $\bar{T}_{3/2}$ and $\hat{T}_{3/2}$ are found by formulas of the constant lapse rates (4.8) applied in each layer separately. Although $\bar{T}_{3/2}$ and $\hat{T}_{3/2}$ are the functions of other parameters, they are listed here in order to simplify comparison between two profiles.

One of the stability conditions for three-time-level schemes can be formulated in the terms of the temperature values: the basic temperature should be warmer then the actual one [4,7,13]. The experiments show that it is not true for two-time-level schemes. For example, the use of the temperature profiles

$$\bar{T}_{5/2}=273\,\text{K},\quad \bar{T}_{3/2}=241.69\,\text{K},\quad \bar{\Gamma}_{5/2}=\bar{\Gamma}_{3/2}=0.006\,\text{K/m},$$

and

$$\hat{T}_{5/2}=273\,\text{K},\quad \hat{T}_{3/2}=240.50\,\text{K},\quad \hat{\Gamma}_{5/2}=0.008\,\text{K/m},\quad \hat{\Gamma}_{3/2}=0.005\,\text{K/m},$$

gives unstable scheme with amplification factor shown in Fig. 3. As it is seen the used implicit treatment of viscosity can recover stability of the two-time-level scheme at least for small time steps if sufficiently great viscosity coefficients are applied ($\alpha=\beta\geqslant 10$). Recall, that respective three-time-level scheme is absolutely stable (see Fig. 4).

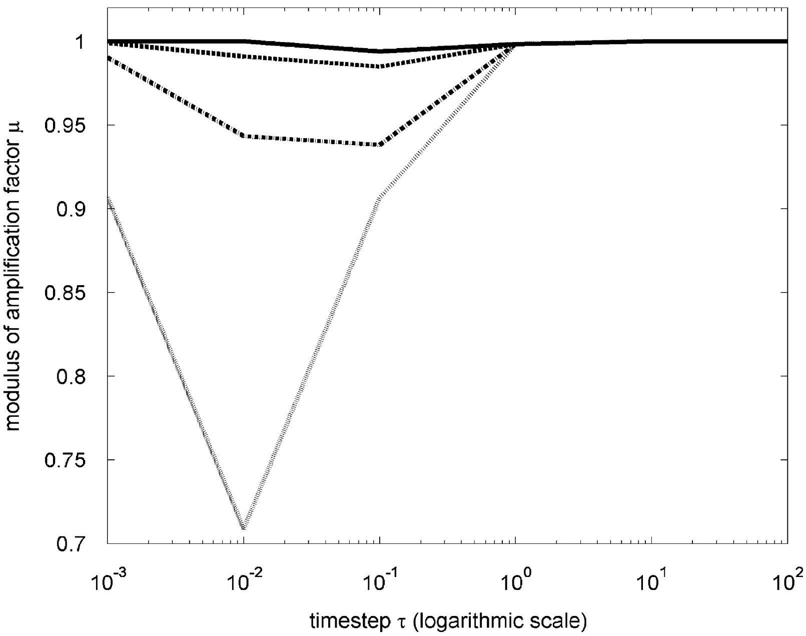

Fig. 2. Absolute stability when (4.10) is violated in some vertical layers.

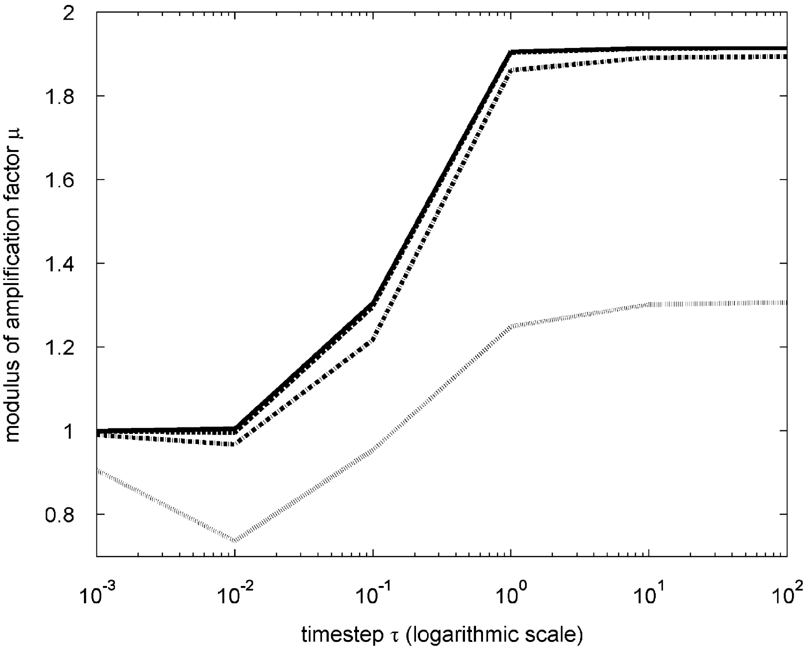

Fig. 3. Absolute instability when the basic temperature is warmer than the actual one.

Conditions (4.6)–(4.7) have been obtained as necessary and sufficient for stability in the cases of large time steps and proportional matrices of the vertical structure. In other cases these conditions can be violated still keeping the stability of the scheme. For example, if the following parameters of the vertical discretization are used

$$\bar{T}_{5/2} = 273\,\mathrm{K}, \quad \bar{T}_{3/2} = 241.69\,\mathrm{K}, \quad \bar{\Gamma}_{5/2} = \bar{\Gamma}_{3/2} = 0.006\,\mathrm{K/m},$$

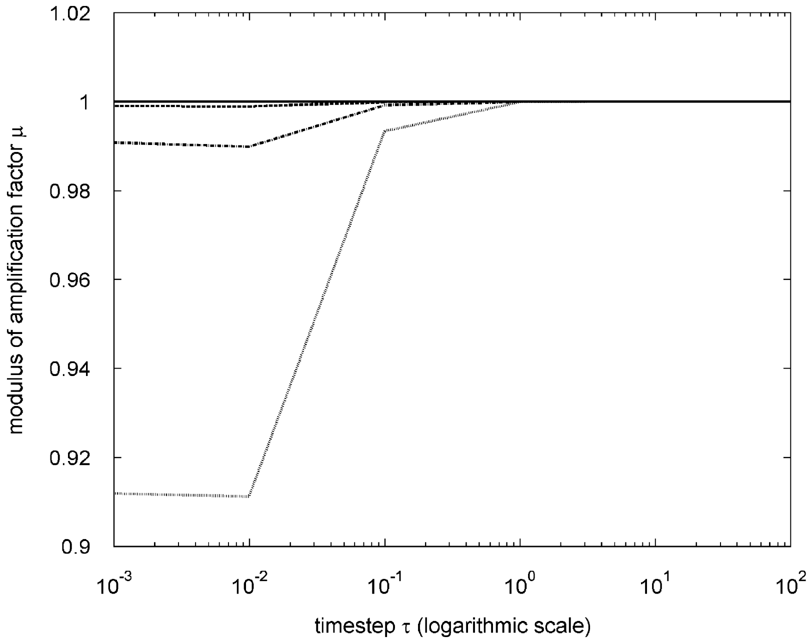

Fig. 4. Absolute stability of three-time-level scheme for the warmer temperature profile.

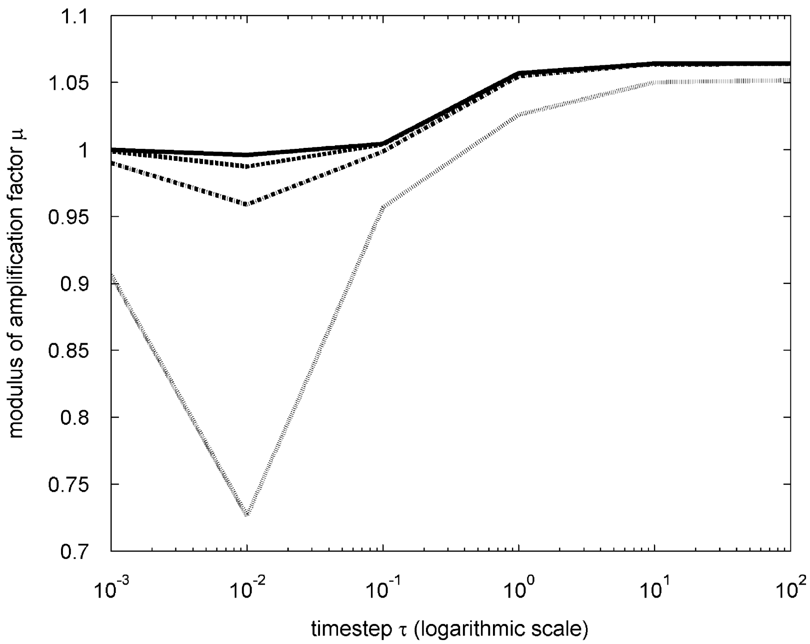

Fig. 5. Stability for small time steps when conditions (4.6)–(4.7) are violated.

and

$$\hat{T}_{5/2} = 273\,\mathrm{K}, \quad \hat{T}_{3/2} = 240.03\,\mathrm{K}, \quad \hat{\Gamma}_{5/2} = 0.004\,\mathrm{K/m}, \quad \hat{\Gamma}_{3/2} = 0.008\,\mathrm{K/m},$$

then condition (4.6) is violated, but the scheme is stable for small values of the time step as it is shown in Fig. 5. Of course, for large time steps the scheme becomes unstable according to the analysis.

The performed numerical experiments with increased vertical resolution (10-level and 30-level vertical grids) showed similar results.

7. Conclusions

In this paper we have performed analytical and numerical study of the stability of two-time-level semi-implicit schemes for the hydrostatic primitive equations. The case of large time steps has been solved analytically providing the stability conditions for arbitrary temperature profiles. In the case of the constant lapse rates, the obtained conditions assume the simple form: the stability is provided if, and only if, the basic lapse rate does not exceed the actual one. Analogous analytical results have been obtained for the case of the proportional reference and actual profiles. The obtained analytical constraints are two times more stringent than those for three-time-level schemes [4,7]. Similar relation between stabilities of two schemes was reported by Simmons and Temperton [18]. It was also noted that in the limiting case of unbounded atmosphere the stability range of three-time-level schemes reduces to the range of two-time-level schemes.

Similar analysis was also made for the modified scheme proposed by Bénard [3] and an increased stability for the actual temperature profiles with positive lapse rates was confirmed. However, in the presence of nearly isothermal or negatively stratified atmospheric layers, we have found that general stability of the modified scheme is very close to stability of the classical one.

Finally, the performed numerical evaluations allowed us to confirm some analytical results and also to reveal some stability properties, which do not appear in the solved analytical cases, but can emerge in more complex situations. It was shown that violation of the derived analytical conditions of stability in some vertical layers of the atmosphere does not necessarily result in an unstable scheme and that a scheme can be stable for small time steps when conditions of stability for large time steps are violated. These experiments also confirmed the stabilizing effect of the viscosity noted in previous studies.

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